Integration 3: The Definite Integral

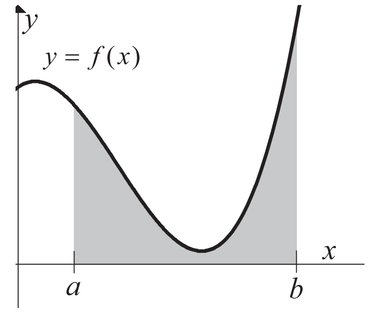
Model 1: The Definite Integral as an Area

As we saw in the last activity, the area (L) of the shaded region of the graph of the positive continuous function below is equal to $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x = L$ where $\Delta x = \dfrac{b-a}{n}$, and x_i^* is any number in $[x_{i-1}, x_i]$, the interval that defines the base of the i^{th} rectangle.

This area, L is equal to a quantity called the **definite integral** of $f(x)$ from a to b, and is represented…

$$\boxed{\int_a^b f(x)\,dx = L}$$

The expression in the box is read…



"***The definite integral of f of x, d-x from x = a to x = b is equal to L.***"

The a and b next to $\int$ (the **integral sign**), are called the lower and upper **limits of integration**.

The integral sign is intended as a script S (for **S**um) to remind us of the connection between the definite integral and Riemann sums. Because this limit L exists, the function is called **integrable**.†

$f(x)$ is called the **integrand**, and dx specifies that the independent variable is x.

† For functions that are not necessarily continuous, the definition of integrability is more subtle and requires the use of partitions of $[a,b]$ such that the subintervals do not necessarily have equal length.

Construct Your Understanding Questions (to do in class)

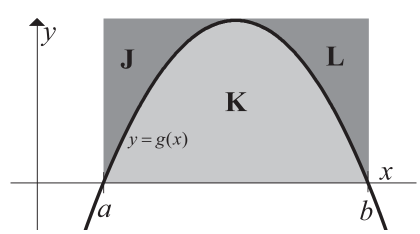
1. For the Riemann sum in Model 1, Δx gives the **height** or **width** [circle one] of each rectangle, and $f(x_i^*)$ gives the **height** or **width** [circle one] of the i^{th} rectangle.

2. What symbol in the box in Model 1 showing the **definite integral** appears to be…
   a. analogous to the symbol Δx found in a Riemann sum?
   b. analogous to the symbol $f(x_i^*)$ found in a Riemann sum?

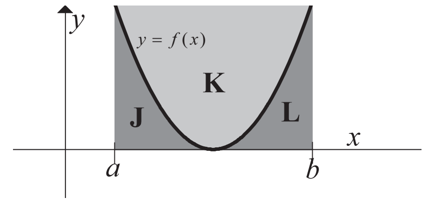
3. Which is equal to $\int_a^b g(x)\,dx$? [circle one]

   Area J   Area K   Area L   Areas J + K

   Areas K + L   Areas J + L   Areas J + K + L



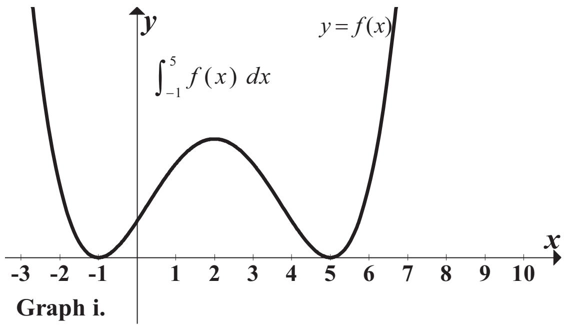
4. Which is equal to $\int_a^b f(x)\,dx$? [circle one]



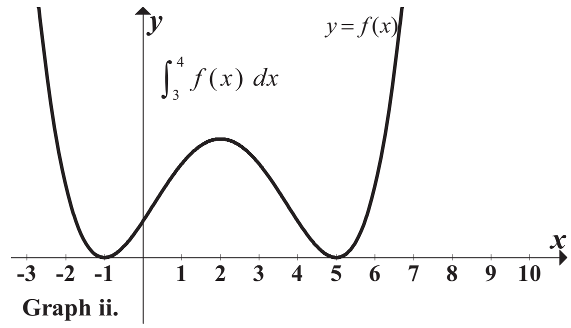
Area J   Area K   Area L   Areas J + K

Areas K + L   Areas J + L   Areas J + K + L

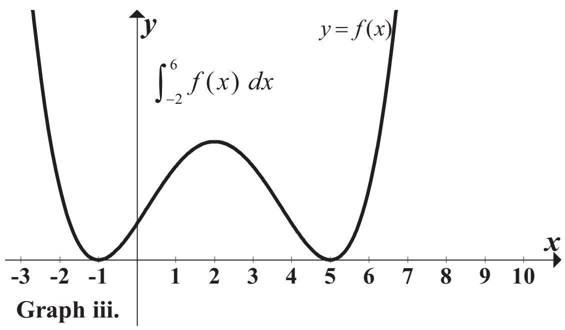
5. For each graph, shade the area equal to the definite integral given.



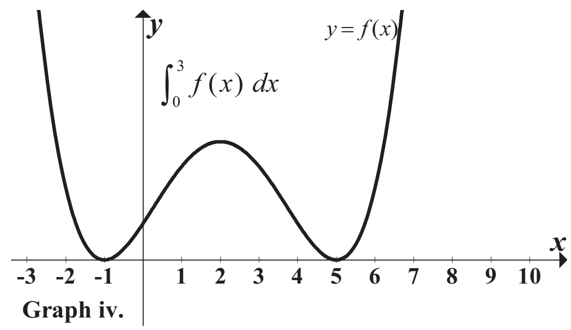
Graph i.



Graph ii.



Graph iii.



Graph iv.

6. **A definite integral is a number**. For each definite integral in the previous question, this number is equal to the area that you shaded. Rank the four definite integrals in the previous question from largest to smallest.

7. For the function shown in Question 5…

   a. Give the numerical value of the definite integral…

      i. $\int_1^1 f(x)\,dx =$ ii. $\int_4^4 f(x)\,dx =$

   b. Write the definite integral from Question 5 that is equal to…
   (It may help to sketch $y = f(x)$ and shade the area equal to each integral.)

      i. $\int_{-1}^{1} f(x)\,dx + \int_1^5 f(x)\,dx =$

      ii. $\int_2^3 f(x)\,dx + \int_0^2 f(x)\,dx =$

8. (Check your work) Are your answers to the previous question consistent with Summary Box I3.1? If not, go back and correct your work.

Summary Box I3.1: Some Properties of Definite Integrals

If a, b, c are real numbers and f is a continuous function:

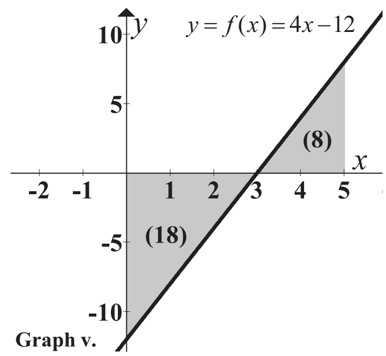
$$\int_a^a f(x)\,dx = 0 \qquad \text{and} \qquad \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

9. For **Graph v**, let x be time in seconds, and $f(x)$ be the velocity (in m/s) of an object that is moving to the left and then to the right.

   Use the areas of the shaded regions to find …

   a. At $x = 5$, how far is the object from where it was at $x = 0$?

   b. At $x = 5$, is the object to the **left** or **right** [circle one] of where it was at $x = 0$?



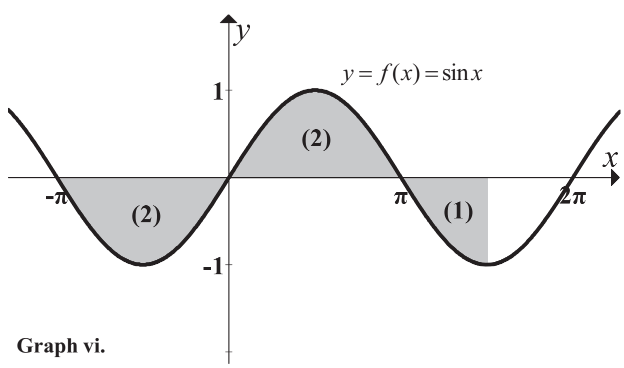
Graph v.

10. (Check your work) For $f(x)$ on **Graph v**, $\int_0^5 f(x)\,dx = -10$. Explain how both parts (a and b) of your answer to the previous question can be combined to give the value of this definite integral. If your answers do not match this value, go back and check your work.

11. For $f(x)$ on **Graph vi**, at right,

    find $\int_{-\pi}^{\frac{3\pi}{2}} f(x)\,dx =$

12. (Check your work) A student gives a value of 5 for the definite integral in the previous question. Explain the error this student is likely making.



Graph vi.

13. Use **Graphs v** and **vi** on this page to find the value of each definite integral.

    a. $\int_0^3 (4x-12)\,dx =$

    b. $\int_3^5 (4x-12)\,dx =$

    c. $\int_2^4 (4x-12)\,dx =$

    d. $\int_0^{\frac{\pi}{2}} \sin x\,dx =$

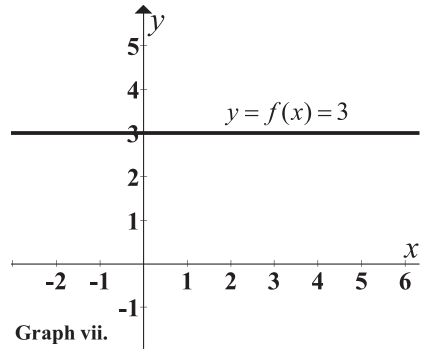
    e. $\int_{\frac{\pi}{2}}^{2\pi} \sin x\,dx =$

    f. $\int_{-\frac{\pi}{2}}^{\pi} \sin x\,dx =$

14. For the constant function $f(x)=3$ shown on **Graph vii**, at right…

    a. Shade an area equal to $\int_2^6 f(x)\,dx$

    b. Calculate the value of $\int_2^6 f(x)\,dx =$



Graph vii.

15. Consider the generalized constant function $f(x)=k$, where k is a constant (e.g., in the previous question $k=3$). In terms of k, a, and b, what is $\int_a^b k\,dx =$

16. (Check your work) For the function $f(x)=8$, the value of $\int_3^7 f(x)\,dx = 32$. Does your formula in the previous question give you this value? If not, check your formula.

17. Use the formula you derived in Question 15 to find the value of each definite integral.

    a. For $f(x)=3$ (shown on **Graph vii**), what is $\int_6^2 f(x)\,dx =$

    b. For $f(x)=8$ what is $\int_7^3 f(x)\,dx =$

18. (Check your work) For $f(x)=3$,
    compare $\int_6^2 f(x)\,dx$ (in Question 17a),
    with $\int_2^6 f(x)\,dx$ (in Question 14),
    and describe how…

    a. the limits of integration differ.

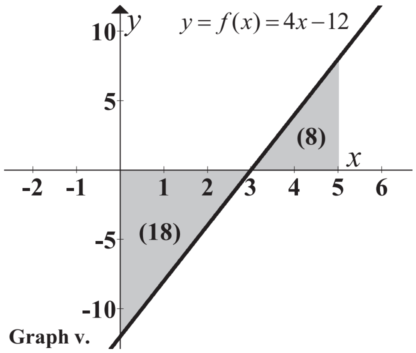
    b. the values differ.

19. Based on what you have discovered so far, what can you say about the value of $\int_b^a k\,dx$ compared to the value of $\int_a^b k\,dx$? (k = a constant)

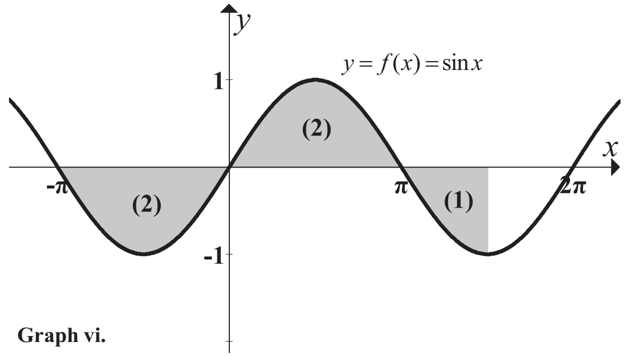
20. Recall that the definite integral, $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$, where $\Delta x = \dfrac{b-a}{n}$.

    a. What is the sign of Δx when $b>a$?

    b. What is the sign of Δx when $b<a$?



Graph v.



Graph vi.

21. Consider the following pairs of definite integrals along with selected values:

$\int_0^5 (4x-12)\,dx = -10$	$\int_0^\pi \sin x\,dx = 2$	$\int_1^4 x^2\,dx =$	$\int_a^b f(x)\,dx = L$
$\int_5^0 (4x-12)\,dx = 10$	$\int_\pi^0 \sin x\,dx = -2$	$\int_4^1 x^2\,dx = -21$	$\int_b^a f(x)\,dx =$

a. Determine the pattern, and guess the two missing values of the integrals above, based on this pattern.

b. Use this pattern and **Graphs v** and **vi** (reprinted above) to find…

i. $\int_3^0 (4x-12)\,dx =$ ii. $\int_\pi^{-\frac{\pi}{2}} \sin x\,dx =$

c. Explain how your answer to Question 20 is related to this pattern.

22. For $f(x) = 3$, shown on **Graph vii** in Question 14, what is…

a. $\int_1^5 f(x)\,dx =$ b. $\int_1^5 7 \cdot f(x)\,dx =$

23. (Check your work) Are your answers to Questions 15, 19, 21 and 22 consistent with Summary Box I3.2? If not, go back and check your work.

Summary Box I3.2: More Properties of Definite Integrals

If a and b are real numbers, k is a constant, and f is a continuous function:

$$\int_a^b k\,dx = k(b-a) \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

24. Use **Graphs viii** and **ix** to find…

a. $\int_0^{2\pi} g(x)\ dx =$

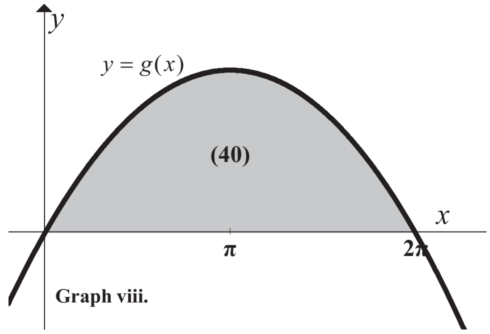
b. $\int_0^{2\pi} 2 \cdot g(x)\ dx =$

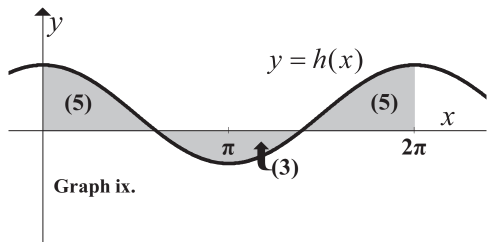
c. $\int_0^{2\pi} h(x)\ dx =$

d. $\int_0^{2\pi} \left[g(x)+h(x)\right]\ dx =$

e. $\int_0^{2\pi} \left[g(x)-h(x)\right]\ dx =$

f. $\int_0^{2\pi} \left[g(x)-3\cdot h(x)\right]\ dx =$





25. (Check your work) Are your answers to the previous question consistent with Summary Box I3.3? If not, go back and correct your work.

Summary Box I3.3: Addition, Subtraction or Multiplication by a Constant

$$\int_a^b [f(x)+g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx \qquad (f \text{ and } g \text{ integrable on } [a,b])$$

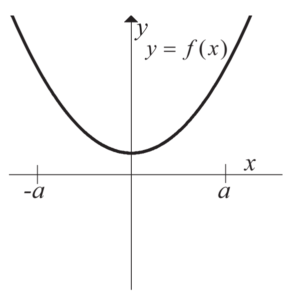
$$\int_a^b [f(x)-g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \qquad (f \text{ and } g \text{ integrable on } [a,b])$$

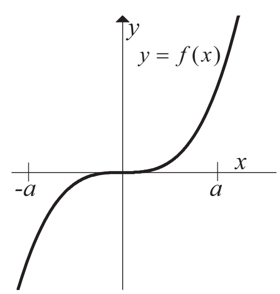
Extend Your Understanding Questions (to do in or out of class)

26. One graph at right shows an even function, the other an odd function.

a. Label each.

b. Shade the area represented by $\int_{-a}^{a} f(x)\ dx$ on each graph.





27. Assume $f(x)$ continuous on $[-a,a]$.

a. Which <u>one</u> of the following statements is true? (Mark the other false.)

i. **True** <u>or</u> **False**: For any even function, $\int_{-a}^{a} f(x)dx = 0$.

ii. **True** <u>or</u> **False**: For any odd function, $\int_{-a}^{a} f(x)dx = 0$.

b. Circle even or odd, and fill in limits of integration that make the statement true .

For an **even** or **odd** [circle one] function, $\int_{-a}^{a} f(x)dx = 2\cdot\int_{\square}^{\square} f(x)dx$

c. (Check your work) Are your answers to this question consistent with your labels and shading in the previous question? If not, check your work.

28. Find $\int_a^b f(x)\,dx$ when… Sketch a graph for each.

a. $a=-1$, $b=4$, $f(x)=x$

b. $a=-1$, $b=4$, $f(x)=\begin{cases} 2 & x\le 2 \\ x & x>2 \end{cases}$

29. Find $\int_a^b f(x)\,dx$ when… Sketch a possible graph for each.

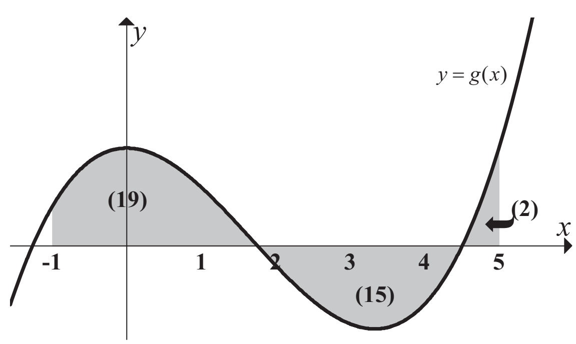
a. $a=0$, $b=5$, and $f(x)=3$

b. $a=3$, $b=1$, and $f(x)=x$

c. $a=b$ and $f(x)$ is any function defined at $x=a$.

d. $a=1$, $b=5$, $\int_1^3 f(x)\,dx=12$, and $\int_3^5 f(x)\,dx=10$
f integrable on $[a,b]$

e. When $a=-1$, $b=5$, and $f(x)=2\cdot g(x)$ (Where $y=g(x)$ is shown on the graph at right.)



30. (Check your work) Each part of the previous question demonstrates a property listed in Summary Boxes I3.1-3. Match each part of the question to the correct property by writing each of the following labels next to the appropriate property in Summary Boxes I3.1-3.

Question 29a, Question 29b, Question 29c, Question 29d, and Question 29e

Notes